P190.

14. Sol:

$y_1'(t)=1,\quad y_1''(t)=0$

$\Rightarrow\ t^2y_1''-t(t+2)y_1'+(t+2)y_1=-t^2-2t+t^2+2t=0.$

$y_2'(t)=e^t+te^t \qquad y_2''(t)=2e^t+te^t$

$\Rightarrow t^2y_2''-t(t+2)y_2'+(t+2)y_2=\left\{t^2(t+2)-t(t+2)(t+1)+(t+2)\right\}e^t=0.$

i.e. y_1 & y_2 are sol's of corresponding homogeneous equ.

set $y(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ to be a particular sol,

then $\begin{cases}u_1'y_1+u_2'y_2=0\\ u_1'y_1'+u_2'y_2'=2t\end{cases} \Rightarrow \begin{pmatrix}u_1'\\u_2'\end{pmatrix}=\dfrac{1}{W(y_1,y_2)}\begin{pmatrix}y_2' & -y_2\\ -y_1' & y_1\end{pmatrix}\begin{pmatrix}0\\2t\end{pmatrix}$

$W(y_1,y_2)=\det\begin{pmatrix}t & te^t\\ 1 & (1+t)e^t\end{pmatrix}=t^2e^t$

$\Rightarrow\quad u_1'(t)=\dfrac{-2t^2e^t}{t^2e^t}=-2,\qquad u_2'(t)=\dfrac{2t^2}{t^2e^t}=2e^{-t}$

$\Rightarrow\quad u_1(t)=-2t \qquad u_2(t)=-2e^{-t}$

$\Rightarrow\ y(t)=-2t^2-2t$ is one particular sol.

15. Sol: (verification omitted)

$W(y_1,y_2)=\det\begin{pmatrix}1+t & e^t\\ 1 & e^t\end{pmatrix}=te^t$

$u_1'(t)=\dfrac{-y_2(t)g(t)}{W(y_1,y_2)}=-\dfrac{e^t\,te^{2t}}{te^t}=-e^{2t}\Rightarrow u_1(t)=-\frac{1}{2}e^{2t}.$

$u_2'(t)=\dfrac{y_1(t)g(t)}{W(y_1,y_2)}=\dfrac{(1+t)te^{2t}}{te^t}=(1+t)e^t\Rightarrow u_2(t)=te^t.$

$\Rightarrow Y(t)=u_1y_1+u_2y_2=-\frac{1}{2}(1+t)e^{2t}+te^{2t}=\frac{1}{2}(t-1)e^{2t}$

P_{226}.

12. Sol: (verification is omitted)

$$W(y_1, y_2, y_3, y_4) = \det \begin{pmatrix} 1 & t & \cos t & \sin t \\ 0 & 1 & -\sin t & \cos t \\ 0 & 0 & -\cos t & -\sin t \\ 0 & 0 & \sin t & -\cos t \end{pmatrix} = \cos^2 t + \sin^2 t = 1$$

14. Sol: (verification omitted)

$$W(y_1, y_2, y_3, y_4) = \det \begin{pmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & (1-t)e^{-t} \\ 0 & 0 & e^{-t} & (t-2)e^{-t} \\ 0 & 0 & -e^{-t} & (3-t)e^{-t} \end{pmatrix}$$

$$= \det \begin{pmatrix} e^{-t} & (t-2)e^{-t} \\ -e^{-t} & (3-t)e^{-t} \end{pmatrix} = e^{-2t}$$

26. Sol:

$$y(t) = y_1(t)\, v(t)$$

$$y'(t) = y_1' v + y v' \quad ; \quad y'' = y_1' v' + y_1'' v + y_1' v' + y_1 v''$$

$$y''' = y_1'' v' + y_1' v'' + y_1''' v + y_1'' v' + y_1'' v' + y_1' v'' + y_1' v'' + y_1 v'''.$$

$$y''' + p_1 y'' + p_2 y' + p_3 y$$

$$= v\, y_1''' + 3v' y_1'' + 3v'' y_1' + \cancel{3}\, v''' y_1$$

$$+ p_1 (v y_1'' + 2v' y_1' + v'' y_1) + p_2 (v y_1' + v' y_1) + p_3 y_1 v$$

$$= y_1 v''' + (3y_1' + p_1 y_1) v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1) v'$$

$$+ \left(y_1''' + p_1 y_1'' + p_2 y_1' + p_3 y_1\right) v$$

$$= y_1 v''' + \left(3\cancel{8} y_1' + p_1 y_1\right) v'' + \left(3y_1'' + 2p_1 y_1' + p_2 y_1\right) v' = 0.$$

27. Sol:

$$e^t \left\{ v''' + \left(3 + \frac{2t-3}{2-t}\right) v'' + \left(3 + \frac{2(2t-3)}{2-t} - \frac{t}{2-t}\right) v' \right\} = 0.$$

set $u := v'$ then

$$u'' + \frac{3-t}{2-t}u' = 0 \quad \Rightarrow \quad u' = c(2-t)e^{-t} = \tilde{c}(t-2)e^{-t} = v''.$$

i.e. $v(t) = c_1 t e^{-t} + c_2 t + c_3$

$\Rightarrow y(t) = v(t)y_1(t) = c_1 t + c_2 t e^t + c_3 e^t.$

P234.

13. Sol:

chara equ: $2\lambda^3 - 4\lambda^2 - 2\lambda + 4 = 0$

$\Leftrightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\Leftrightarrow (\lambda-1)(\lambda+1)(\lambda-2) = 0,$

i.e. $\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 2.$

$\Rightarrow y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}$

22. Sol:

chara equ: $\lambda^4 + 2\lambda^2 + 1 = 0$

$\Leftrightarrow (\lambda^2+1)^2 = 0$

i.e. $\lambda_1 = \lambda_2 = i \qquad \lambda_3 = \lambda_4 = -i.$

$\Rightarrow y(t) = (a_0 + a_1 t)\cos t + (b_0 + b_1 t)\sin t.$

P245.

16. Sol:

chara equ: $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \Leftrightarrow (\lambda-1)^3 = 0$

i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 1.$

$y_1 = e^t, \qquad y_2 = te^t \qquad y_3 = t^2 e^t.$

$y_1' = e^t \qquad y_2' = (1+t)e^t \qquad y_3' = (2t+t^2)e^t$

$y_1'' = e^t \qquad y_2'' = (2+t)e^t \qquad y_3'' = (t^2+4t+2)e^t.$

$$Y(t) = u_1 y_1 + u_2 y_2 + u_3 y_3$$

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$$

i.e. $$\begin{pmatrix} e^t & te^t & t^2 e^t \\ e^t & (t+1)e^t & (t^2+2t)e^t \\ e^t & (t+2)e^t & (t^2+4t+2)e^t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$$

set $$A := e^t \begin{pmatrix} 1 & t & t^2 \\ 1 & t+1 & t^2+2t \\ 1 & t+2 & t^2+4t+2 \end{pmatrix} =: e^t B$$

then $A^{-1} = e^{-t} B^{-1}$

$$(B \mid I) = \left(\begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 1 & t+1 & t^2+2t & 0 & 1 & 0 \\ 1 & t+2 & t^2+4t+2 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 0 & 1 & 2t & -1 & 1 & 0 \\ 0 & 2 & 4t+2 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 0 & 1 & 2t & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 0 & 1 & 2t & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & t & 0 & 1-\frac{1}{2}t^2 & t^2 & -\frac{1}{2}t^2 \\ 0 & 1 & 0 & -1-t & 1+2t & -t \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1+t+\frac{1}{2}t^2, & -t-t^2, & \frac{1}{2}t^2 \\ 0 & 1 & 0 & -1-t, & 1+2t, & -t \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

i.e. $$A^{-1} = e^{-t} \begin{pmatrix} 1+t+\frac{1}{2}t^2 & -(t^2+t) & \frac{1}{2}t^2 \\ -(1+t) & 1+2t & -t \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = e^{-t} \begin{pmatrix} 1+t+\frac{1}{2}t^2 & -(t^2+t) & \frac{1}{2}t^2 \\ -(1+t) & 1+2t & -t \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$$

$$= e^{-t} \begin{pmatrix} \frac{1}{2}t^2 g \\ -tg \\ \frac{1}{2} g \end{pmatrix}$$

$$\Rightarrow u_1 = \int \frac{1}{2} t^2 e^{-t} g \, dt$$

$$u_2 = -\int t e^{-t} g \, dt$$

$$u_3 = \int \frac{1}{2} e^{-t} g \, dt$$

i.e. $Y = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$= \int_{t_0}^{t} \frac{1}{2} s^2 e^{-s} e^{t} g(s) \, ds$$

$$- \int_{t_0}^{t} s e^{-s} t e^{t} g(s) \, ds$$

$$+ \int_{t_0}^{t} \frac{1}{2} e^{-s} t^2 e^{t} g(s) \, ds$$

$$= \frac{1}{2} \int_{t_0}^{t} (t-s)^2 e^{t-s} g(s) \, ds .$$

if $g(t) = t^{-2} e^{t}$, then

$$Y(t) = \frac{1}{2} \int_{t_0}^{t} (t-s)^2 e^{t-s} s^{-2} e^{s} \, ds$$

$$= \frac{1}{2} e^{t} \int_{t_0}^{t} \left(\frac{t-s}{s} \right)^2 ds$$

$$= \frac{1}{2} e^{t} \left\{ - \frac{t^2}{s} \Big|_{t_0}^{t} - 2t \log|s| \Big|_{t_0}^{t} + s \Big|_{t_0}^{t} \right\}$$

$$= - t e^{t} \log|t| + c .$$